INTRODUCTION TO THE LAPLACE TRANSFORM

CHRISTOPHER COPPOLA

Although operational mathematics has been in existence for almost two hundred years, the first systematic development of this procedure was made by Oliver Heaviside, in his application of the differential operator to solving differential equations that governed electromagnetic theory. As an engineer, he frequently encountered discontinuous terms that were not easily compatible with previous differential methods; eventually the function modeling them would be named after him:

$$u_b(t) = \begin{cases} 0, & \text{if } t < b \\ 1, & \text{if } t \geq b \end{cases}$$

The function $u_b(t)$ is called the Heaviside function. Although it occurs often in electrical circuits, this function can be found in many places in both science and mathematics. The advantage to using this function is that when integrated with the Laplace transform, a continuous function is obtained, and the differential equation can be solved. Heaviside used integral transforms to simplify the process of solving ordinary differential equations. Eventually his methods would be superceded by the more rigorous formulation presented by Laplace, but his applications and development of them were important and original.

The most frequent application of the Laplace transform is in engineering and physics. Beginning with any linear differential equation, the Laplace transform can be applied to each term of the equation, transforming all of the different derivatives into an equation with no derivatives. Using complex algebraic methods, this equation can be solved in terms of the transform of the original function. Then the inverse transform is applied to the remaining terms of the equation, and the result is that the original function is solved for, and no differential methods have been applied. This can greatly simplify the process of solving differential equations, which is why it is so widely used in engineering and scientific applications.

One of many integral transforms used in mathematics, the Laplace transform corresponds to the kernel:

$$e^{-st}$$

Exponential functions of this form occur frequently in differential equations. The Laplace transform has many interesting applications in mathematics, and one of the more common uses is to solve linear differential equations. The Laplace transform changes the operations of calculus into those of algebra, sometimes making solutions significantly easier to obtain.

If the parameter s is changed to $-i\omega$, the kernel

$$e^{iwt}$$

is obtained. This is the kernel for the familiar Fourier transform, which is also used frequently in mathematics.

The Laplace transform is only defined under certain conditions. If a function

$$f : t \to \mathbb{R}, t \in \mathbb{R}$$

is defined for all positive values of t, then the Laplace transform of $f(t)$, $F(s)$, is defined by

$$F(s) = \int_0^\infty e^{-st} f(t)\, dt$$

if this integral exists. Some care must be taken in evaluating the transform of certain functions due to the improper integral. In order to obtain some useful operations with which to solve differential equations, additional conditions must be specified on $f(t)$. The function must be piecewise continuous, so that the integral over the sum of the intervals may be replaced by the sum of the integrals over each interval. The function $f(t)$ must also be of exponential order -

$$e^{-\alpha t}|f(t)| < M$$
$$e^{-st}|f(t)| < Me^{(\alpha - s)t}$$

- so that the Laplace integral converges when $s > \alpha$. Conditions on the continuity of the derivatives of $f(t)$ depend on the order of the derivatives being transformed. The Laplace transform of $f(t)$ is generally abbreviated as:

$$\mathcal{L}\{f(t)\} = F(s)$$

The Laplace transform must be linear, since by definition it is integration with respect to a kernel:

$$\mathcal{L}\{f(t)+g(t)\} = \int_0^\infty [f(t)+g(t)]e^{-st}\,dt = \int_0^\infty f(t)e^{-st}\,dt + \int_0^\infty g(t)e^{-st}\,dt = \mathcal{L}\{f(t)\}+\mathcal{L}\{g(t)\}$$

The transform of a sum of functions is just the sum of the transforms of the functions. The transform of a series is then a series of transforms. This property is very useful in solving linear differential equations.

Calculating the Laplace transform of some simpler functions will allow us to calculate more complicated ones, and also derive the inverse Laplace transform for some functioncs necessary to solve differential equations. If $f(t) = 1$, we have

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st}\,dt = -\frac{1}{s}\,e^{-st}\Big]_0^\infty$$

For $s > 0$, we can define the Laplace transform of 1:

$$\mathcal{L}\{1\} = \frac{1}{s}$$

If $f(t) = t$, we have

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty te^{-st}\,dt = -\frac{1}{s}\,te^{-st}\Big]_0^\infty + \frac{1}{s}\int_0^\infty e^{-st}\,dt = -\frac{1}{s^2}\,e^{-st}\Big]_0^\infty = \frac{1}{s^2}$$

Again, for $s > 0$, we can define the Laplace transform of t to be:

$$\mathcal{L}\{t\} = \frac{1}{s^2}$$

In a similar way, the Laplace transform of any function t^α can be calculated. A general formula for this transform exists, given that $s > 0$ and $\alpha > -1$:

$$\mathcal{L}\{t^\alpha\} = \frac{\Gamma(\alpha+1)}{s^{\alpha+1}}$$

One of the most important properties of the Laplace transform is the transform of the derivative. It is possible to express the transform of the derivative of a function f(t) in terms of the transform of the original function:

$$\mathcal{L}\{f'(t)\} = \int_0^\infty f'(t)e^{-st}\,dt = f(t)e^{-st}\Big]_0^\infty + \int_0^\infty sf(t)e^{-st}\,dt = -f(0) + s\mathcal{L}\{f(t)\}$$

$$\mathcal{L}\{f'(t)\} = sF(s) - f(0)$$

This formula can be iterated to obtain the transforms of higher derivatives as well:

$$\mathcal{L}\{f''(t)\} = s\mathscr{L}\{f'(t)\} - f'(0) = s^2F(s) - sf(0) - f'(0)$$

$$\mathcal{L}\{f^n(t)\} = s^nF(s) - s^{n-1}f(0) - s^{n-2}f'(0) - ... - f^{n-1}(0)$$

It is obvious that applying the Laplace transform to differential equations can potentially greatly simplify the process of solving them. After transforming the derivatives into products, isolating the transform of the function is very simple. However, applying the inverse Laplace transform to solve for the function can be difficult. The inverse Laplace transform can be denoted as follows:

$$F(s) = \mathcal{L}\{f(t)\}$$

$$\mathcal{L}^{-1}F(s) = \mathcal{L}^{-1}\mathcal{L}\{f(t)\} = f(t)$$

This is simply the inverse of the forward transform. For the purposes of solving most differential equations, it is not necessary to define and evaluate explicitly the inverse Laplace transform; rather, simply applying the known transforms and their inverses (obtained by transforming functions in the forward direction) leads to all the necessary inverse transforms.

A powerful operation that can greatly assist in simplifying the process of solving the inverse transform is convolution. This operation provides a way to find the inverse transform of a product of functions without evaluating the inverse transform explicitly. Let $f(t)$ and $g(t)$ be piecewise continuous functions of exponential order for $0 \leq t < \infty$. Then, for $s > \alpha$:

$$F(s) = \int_0^\infty f(t)e^{-st}\,dt$$

$$F(s)e^{-bs} = \int_0^\infty f(t)e^{-st}e^{-bs}\,dt = \int_0^\infty f(t)e^{-s(t+b)}\,dt$$

A substitution will allow this function to be re expressed in terms of the Laplace transform:

$$\tau = t + b$$

$$\int_b^\infty f(\tau - b)e^{-s\tau}\,d\tau = \int_0^b 0\,d\tau + \int_b^\infty f(\tau - b)e^{-s\tau}\,d\tau$$

Define a function $u_b(t)$ such that

$$u_b(t) = \begin{cases} 0, & \text{if } t < b \\ 1, & \text{if } t \geq b \end{cases}$$

The function $u_b(t)$ is the Heaviside function. Incorporating this into our integral, we may rewrite the previous equality:

$$F(s)e^{-bs} = \int_0^\infty u_b(t)f(\tau)e^{-s\tau}\,d\tau$$

More specifically:

$$F(s)e^{-bs} = \mathcal{L}\{u_b(t)f(t - b)\}$$

We can use this equality to derive the properties of convolution. The Laplace transforms of $f(t)$ and $g(t)$ are defined as:

$$F(s) = \mathcal{L}\{f(t)\}$$
$$G(s) = \mathcal{L}\{g(t)\}$$

Define $g(t)$ to be 0 when $t < 0$. Then, for $s > \alpha$,

$$G(s)e^{-s\tau} = \mathcal{L}\{g(t-\tau)\} = \int_0^\infty e^{-st} g(t-\tau)\,dt$$

$$F(s)G(s) = \int_0^\infty f(\tau)e^{-s\tau}G(s)\,d\tau = \int_0^\infty f(\tau)\int_0^\infty e^{-st}g(t-\tau)\,dt\,d\tau$$

Since $f(t)$ and $g(t)$ are both of exponential order, this integral is uniformly convergent, and the order of integration may be interchanged.

$$\int_0^\infty e^{-st}\int_0^t f(\tau)g(t-\tau)\,d\tau\,dt$$

The integral

$$\int_0^t f(\tau)g(t-\tau)\,d\tau$$

is called the convolution of $f(t)$ and $g(t)$, and is denoted by

$$f(t) * g(t) = \int_0^t f(\tau)g(t-\tau)\,d\tau$$

The previous equality may be rewritten in terms of convolution:

$$F(s)G(s) = \int_0^\infty e^{-st}\int_0^t f(\tau)g(t-\tau)\,d\tau\,dt = \int_0^\infty e^{-st}[f(t)*g(t)]\,dt = \mathcal{L}\{f(t)*g(t)\}$$

We can now define the original functions in terms of the inverse Laplace transform and convolution:

$$f(t) * g(t) = \mathcal{L}^{-1}\{F(s)G(s)\}$$

To fully take advantage of this equality, it is useful to define the transforms of some common functions. Consider the substitution $\tau = at$:

$$\mathcal{L}\{f(\tau)\} = \mathcal{L}\{f(at)\} = \int_0^\infty e^{-st}f(at)\,dt = \frac{1}{a}\int_0^\infty e^{-(\frac{s}{a})\tau}f(\tau)\,d\tau = \frac{1}{a}\,F\left(\frac{s}{a}\right)$$

$$f(at) = \mathcal{L}^{-1}\left\{\frac{1}{a}\,F\left(\frac{s}{a}\right)\right\}$$

The transform of exponential functions is defined as:

$$\mathcal{L}\{e^{-at}\} = \int_0^\infty e^{-at}e^{-st}\,dt = \int_0^\infty e^{-(a+s)t}\,dt = -\frac{1}{s+a}\left[e^{-(a+s)t}\right]_0^\infty = \frac{1}{s+a}$$

$$e^{-at} = \mathcal{L}^{-1}\left\{\frac{1}{s+a}\right\}$$

The Laplace transform can be used to solve differential equations that would otherwise be either more complex or more time consuming to solve using standard differential methods. The Laplace transform also handles difficult functions such as the dirac delta function in a very smooth manner. Let us apply the Laplace transform to one such differential equation:

$$f''(t) + 4f(t) = u_\pi(t)sin(2t), \quad y(0) = 0, \quad y'(0) = 1$$

Solving a differential equation with the Laplace transform requires three different steps. The first step is to take the Laplace transform of both sides, which means the transform of each term since it is a linear operator:

$$\mathcal{L}\{f''(t)\} + 4\mathcal{L}\{f(t)\} = \mathcal{L}\{u_\pi(t)sin(2t)\}$$

$$s^2F(s) - sf(0) - f'(0) + 4F(s) = \int_0^\infty u_\pi(t)sin(2t)e^{-st}\,dt = \int_\pi^\infty sin(2t)e^{-st}\,dt$$

Here we can make a substitution before utilizing integration by parts, twice. Let $\tau = t - \pi$:

$$\int_\pi^\infty sin(2t)e^{-st}\,dt = \int_0^\infty sin(2(\tau+\pi))e^{-s(\tau+\pi)}\,dt = e^{-s\pi}\int_0^\infty sin(2(\tau+\pi))e^{-s\tau}\,dt$$

$$\int_0^\infty sin(2(\tau+\pi))e^{-s\tau}\,dt = -\frac{1}{s}e^{-s\tau}sin(2(\tau+\pi))\Big]_0^\infty + \frac{2}{s}\int_0^\infty cos(2(\tau+\pi))e^{-s\tau}\,dt$$

$$= \frac{2}{s}\left\{-\frac{1}{s}e^{-s\tau}cos(2(\tau+\pi))\Big]_0^\infty - \frac{2}{s}\int_0^\infty sin(2(\tau+\pi))e^{-s\tau}\,dt\right\} = \frac{2}{s^2} - \frac{4}{s^2}\int_0^\infty sin(2(\tau+\pi))e^{-s\tau}\,dt$$

$$\int_0^\infty sin(2(\tau+\pi))e^{-s\tau}\,dt = \frac{2}{s^2+4}$$

The next step is to solve for the transform of the solution:

$$s^2F(s) - sf(0) - f'(0) + 4F(s) = (s^2+4)F(s) - 1 = \frac{2}{s^2+4}e^{-s\pi}$$

$$\mathcal{L}\{f(t)\} = F(s) = \frac{2e^{-s\pi}}{(s^2+4)^2} + \frac{1}{s^2+4}$$

Now that we have solved for the transform of the solution to the differential equation, we must apply the inverse transform to find the solution itself:

$$f(t) = \mathcal{L}^{-1}\left\{\frac{1}{s^2+4}\right\} + \mathcal{L}^{-1}\left\{\frac{2e^{-s\pi}}{(s^2+4)^2}\right\} = \frac{1}{2}sin(2t) + \mathcal{L}^{-1}\left\{\frac{2e^{-s\pi}}{(s^2+4)^2}\right\}$$

In order to evaluate the right hand term, we must utilize the convolution equality:

$$\mathcal{L}^{-1}\left\{\frac{2e^{-s\pi}}{(s^2+4)^2}\right\} = \mathcal{L}^{-1}\left\{e^{-s\pi}\right\} * \mathcal{L}^{-1}\left\{\frac{2}{(s^2+4)^2}\right\} = \delta_\pi(t) * \frac{1}{8}\Big[sin(2t) + 2tcos(2t)\Big]$$

$$= \int_0^t \delta_\pi(t-u)\frac{1}{8}\Big[sin(2t) + 2tcos(2t)\Big]\,du = \frac{1}{8}\Big[sin(2(t-u)) + 2tcos(2(t-u))\Big]_0^t$$

$$= -\frac{1}{8}[sin(2t) + 2tcos(2t)]$$

Finally, we can express the solution as a function of t, without ever having used DE methods:

$$f(t) = \frac{3}{8}sin(2t) - \frac{t}{4}cos(2t)$$

WORKS CITED

Churchill, Operational Mathematics, (McGraw-Hill, York, PA, 1958), 1-36.

Holl, Maple, Vinograde, Intro. to the Laplace Transform, (Appleton-Century-Crofts, New York, 1959), 1-13.

Blanchard, Devaney, Hall, Differential Equations, (Brooks/Cole, Pacific Grove, CA, 2002), 541-602.